Extra Practice Problems 1

This handout contains a bunch of practice problems you can use to improve your skills and generally get prepped for the upcoming midterm exam. As always, you are encouraged to ask questions if you have them – we're happy to help out!

Problem One: First-Order Logic

Here's some more practice with translating statements into first-order logic. Each of these logic translations appeared on some past midterm exam.

  i. Given the predicates

> *Person*(p), which states that p is a person, and
> *ParentOf*(p_1, p_2), which states that p_1 is the parent of p_2,

  write a statement in first-order logic that says "someone is their own grandparent." (Paraphrased from an old novelty song.)

  ii. Given the predicates

> *Set*(S), which states that S is a set, and
> $x \in y$, which states that x is an element of y,

  write a statement in first-order logic that says "for any sets S and T, the set $S \Delta T$ exists."

  iii. Given the predicates

> *Set*(S), which states that S is a set;
> $x \in y$, which states that x is an element of y;
> *Natural*(n), which states that n is a natural number; and
> $x < y$, which states that x is less than y,

  write a statement in first-order logic that says "if S is a nonempty subset of $\mathbb{N}$, then S contains a natural number that's smaller than all the other natural numbers in S." (This is called the ***well-ordering principle***.)

Problem Two: First-Order Negations

For each of the statements you came up with in part (i) of this problem, negate that statement and push the negation as deep as possible, along the lines of what you did in Problem Set Two. Then, for each statement, translate it back into English and make sure you see why it's the negation of the original formula.

Problem Three: Properties of Sets

Below are a number of claims about sets. For each claim, decide whether the statement is true or false. If it's true, prove it. If it's false, disprove it.

    i. For all sets A and B, the following is true: $(A - B) \cup B = A$.

    ii. For all sets A, B, and C, if $A \subseteq B \cap C$, then $A \subseteq B$ and $A \subseteq C$.

    iii. For any set A, if $A \subsetneq \emptyset$, then $137 \in A$.

Problem Four: Latin Squares

A **_Latin square_** is an $n \times n$ grid such that every natural number between 1 and n, inclusive, appears exactly once on each row and column. A **_symmetric Latin square_** is a Latin square that is symmetric across the main diagonal from the upper-left corner to the lower-right corner. Specifically, the elements at positions (i, j) and (j, i) are always the same.

Prove that in any $n \times n$ symmetric Latin square, where n is even, there is at least one number between 1 and n that appears nowhere on the diagonal.

Problem Five: The Logic of Elections

Two candidates X and Y are running for office and are counting final votes. Candidate X argues that more people voted for them than for Candidate Y by making the following claim: "For every ballot cast for Candidate Y, there were two ballots cast for Candidate X." Candidate X states this in logic as follows:

$$\forall b.\ (BallotForY(b) \rightarrow$$
$$\exists b_1.\ \exists b_2.\ (BallotForX(b_1) \wedge BallotForX(b_2) \wedge b_1 \neq b_2)$$
$$)$$

However, it is possible for the above first-order logic statement to be true even if Candidate X didn't get the majority of the votes.

Give an example of a set of ballots such that

    1. every ballot is cast for exactly one of Candidate X and Candidate Y,

    2. the set of ballots obeys the rules described by the above statement in first-order logic, but

    3. candidate Y gets strictly more votes than Candidate X.

You should justify why your set of ballots works, though you don't need to formally prove it. Make specific reference to the first-order logic statement in your justification.

Problem Six: The Minkowski Sum

If $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$, then the **_Minkowski sum_** of A and B, denoted $A + B$, is the set

$$A + B = \{\ m + n \mid m \in A \text{ and } n \in B\ \}$$

This question explores properties of the Minkowski sum.

    i. Prove or disprove: $|A + B| = |A| \cdot |B|$ for all finite sets $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$.

    ii. What is $\mathbb{N} + \mathbb{N}$? Prove it.

Problem Seven: Tournaments

Recall from Problem Set Two that **tournament** is a contest among n players. Each player plays a game against each other player, and either wins or loses the game (let's assume that there are no draws). We can visually represent a tournament by drawing a circle for each player and drawing arrows between pairs of players to indicate who won each game.

For example, in the tournament to the left, player A beat player E, but lost to players B, C, and D. A **tournament winner** is a player in a tournament who, for each other player, either won her game against that player, or won a game against a player who in turn won his game against that player (or both). In the tournament to the left, players B, C, and E are tournament winners. However, player D is not a tournament winner, because he neither won against player C, nor won against anyone who in turn won against player C. Although player D won against player E, who in turn won against player B, who then won against player C, under our definition player D is not a tournament winner.
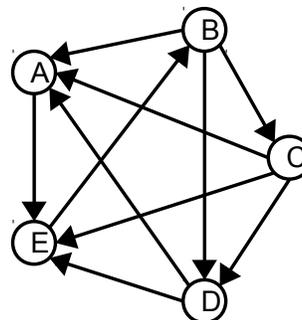
i. Let n be an arbitrary odd natural number. An **egalitarian tournament** with $n = 2k + 1$ players is one where every player won exactly k games. Prove that every player in an egalitarian tournament is a tournament winner

ii. If T is a tournament and p is a player in T, then let $W(p) = \{\ q \mid q$ is a player in T and p beat $q\ \}$. Prove that if p_1 and p_2 are players in T and $p_1 \neq p_2$, then $W(p_1) \neq W(p_2)$.

iii. Is your result from part (ii) still valid if T is just a *pseudotournament*, rather than a full tournament?

Problem Eight: Hungry Logic

(From the Fall 2015 midterm exam)

Let's imagine that you're *really* hungry and want to build an infinitely tall cheese sandwich. Your sandwich will consist an infinite alternating sequence of slices of bread and slices of cheese.

Using the predicates

> *Bread*(b), which states that b is a piece of bread;
> *Cheese*(c), which states that c is a piece of cheese; and
> *Atop*(x, y), which says that x is directly on top of y,

write a statement in first-order logic that says "every piece of bread has a piece of cheese directly on top of it, every piece of cheese has a piece of bread directly on top of it, and there's at least one piece of bread."

Problem Nine: Propositional Logic

Below are a series of English descriptions of relations among propositional variables. For each description, write a propositional formula that precisely encodes that relation. Then, briefly explain the intuition behind your formula. You may find the online truth table tool useful here.

    i. For the variables a, b, c, and d: the variables, written out in alphabetical order, alternate between true and false.

    ii. For the variables a, b, c, and d: the variables, written out in alphabetical order, alternate between true and false, except that your formula cannot use the $\vee$ connective.

Problem Ten: More Modular Arithmetic

Here's a few more properties of the modular congruence relation to explore! In this problem, assume all variables represent integers.

    i. Prove that if $w \equiv_k y$ and $x \equiv_k z$, then $w + x \equiv_k y + z$.

    ii. Prove that if $w \equiv_k y$ and $x \equiv_k z$, then $wx \equiv_k yz$.

Problem Eleven: Rational Roots

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$. A **root** of this equation is a real number x such that $ax^2 + bx + c = 0$. For example, the quadratic equation $x^2 - 3x + 2 = 0$ has 1 and 2 as roots.

Here's an interesting fact: if a, b, and c are odd integers, then $ax^2 + bx + c = 0$ cannot have any roots that are rational numbers. Surprisingly, one of the easiest ways to prove this result is to use properties of odd and even numbers.

Prove that if a, b, and c are odd integers, then $ax^2 + bx + c = 0$ has no rational roots. Although you're probably tempted to pull out the quadratic formula here, we recommend that you *not* do this. Instead, look at what happens if you plug $x = {}^p/_q$ into the formula $ax^2 + bx + c = 0$, and consider the possible parities for p and q (the **parity** of a number is whether it's even or odd).

As a hint, 0 is even.